

Optimization and Decision Making 2018 Examination: Solutions

1

a) If $\sum_{i=1}^n w_i < 1$ then the remaining $1 - \sum_{i=1}^n w_i$ is invested in the risk free asset. If $\sum_{i=1}^n w_i > 1$ then this value is negative, corresponding to an extra $\sum_{i=1}^n w_i - 1$ borrowed at interest rate ρ_0 .

b) We wish to minimise

$$\frac{1}{2}f(w_1, \dots, w_n) = \sum_{i,j=1}^n w_i w_j \sigma_{ij},$$

subject to

$$g(w_1, \dots, w_n) = \sum_{i=1}^n w_i \mu_i + \rho_0(1 - \sum_{i=1}^n w_i) - \mu_{P+} = 0,$$

where μ_{P+} is the target rate of return, including investment in the risk free asset.

We thus consider the Lagrangian

$$L(w_1, \dots, w_n | \lambda_1, \lambda_2) = f(w_1, \dots, w_n) - \lambda g(w_1, \dots, w_n),$$

and differentiate with respect to $w_1, \dots, w_n$ and λ .

For $n = 2$ we have

$$\frac{\partial L}{\partial w_1} = w_1 \sigma_1^2 + w_2 \sigma_{12} - \lambda(\mu_1 - \rho_0) = 0, \quad (1)$$

$$\frac{\partial L}{\partial w_2} = w_2 \sigma_2^2 + w_1 \sigma_{12} - \lambda(\mu_2 - \rho_0) = 0, \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = -w_1 \mu_1 - w_2 \mu_2 - \rho_0(1 - w_1 - w_2) + \mu_{P+} = 0. \quad (3)$$

Thus we have the three equations as required.

c) We thus have two equations in two unknowns. For the values given we have:

$$0.2w_1 + 0.05w_2 - 0.08\lambda = 0 \Rightarrow 4w_1 + w_2 - 1.6\lambda = 0, \quad (4)$$

$$0.05w_2 + 0.05w_1 - 0.04\lambda = 0 \Rightarrow w_1 + w_2 - 0.8\lambda = 0, \quad (5)$$

$$-0.1w_1 - 0.06w_2 - 0.02(1 - w_1 - w_2) + 0.08 = 0 \Rightarrow -8w_1 - 4w_2 + 6 = 0 \quad (6)$$

Taking 2 multiplied by (5) from (4) gives $2w_1 - w_2 = 0$. This in turn substituted into (6) gives $-16w_1 + 6 = 0 \Rightarrow w_1 = 3/8$ and so $w_2 = 3/4$. We thus borrow $3/8 + 3/4 - 1 = 1/8$ from the bank.

Alternatively we can start with the following, and then solve: equation (3) rearranges to

$$w_1(\rho_0 - \mu_1) + w_2(\rho_0 - \mu_2) = \rho_0 - \mu_{P+}.$$

Taking $(\mu_1 - \rho_0)$ times Equation (2) from $(\mu_2 - \rho_0)$ times Equation (1) gives

$$w_1[(\mu_2 - \rho_0)\sigma_1^2 - (\mu_1 - \rho_0)\sigma_{12}] +$$

$$w_2[(\mu_2 - \rho_0)\sigma_{12} - (\mu_1 - \rho_0)\sigma_2^2] = 0.$$

d) In this case, we still have to find the solutions to equations (4-6). Changing the target does not affect equations (4) and (5) at all, and consequently these still lead to $w_2 = 2w_1$. This can then be substituted into equation (6), which will then give a different value for w_1 as a result. A higher target rate of return would increase how much was borrowed from the bank, and a lower target would reduce it (to the point where some money is deposited in the bank). The relative size of the investments in assets 1 and 2, though, will remain constant at 1 : 2.

2

a) Minimise $15x_{11} + 9x_{12} + 6x_{13} + 8x_{14} + 9x_{21} + 11x_{22} + 8x_{23} + 13x_{24} + 5x_{31} + 17x_{32} + 13x_{33} + 16x_{34}$

subject to

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 40,$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 60,$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 100,$$

$$x_{11} + x_{21} + x_{31} \geq 30,$$

$$x_{12} + x_{22} + x_{32} \geq 50,$$

$$x_{13} + x_{23} + x_{33} \geq 50,$$

$$x_{14} + x_{24} + x_{34} \geq 60,$$

$$x_{ij} \geq 0.$$

b) To deal with the difference between the total supply and the total demand, we create a dummy supermarket with demand of 10 (the value of supply minus demand) where all of the transportation costs are 0.

	1	2	3	4	5	
A	15	9	6	8	0	40
B	9	11	8	13	0	60
C	5	17	13	16	0	100
	30	50	50	60	10	

c) Using the least cost algorithm we get the following allocation

	1	2	3	4	5	
A	15	9	6 40	8	0	40
B	9	11 50	8 10	13	0	100
C	5 30	17 0	13	16 60	0 10	60
	30	50	50	60	10	

Setting $u_3 = 0$ we get $v_1 = 5, v_2 = 17, v_4 = 16, v_5 = 0, u_2 = 6, v_3 = 14, u_1 = 8$. This is not optimal since $v_3 - u_3 = 14 > 13$. Thus we add to this square (0 is the maximum!), and adjust the three included squares that form a square with it, to give the following allocation

	1	2	3	4	5	
A	15	9	6 40	8	0	40
B	9	11 50	8 10	13	0	100
C	5 30	17	13 0	16 60	0 10	60
	30	50	50	60	10	

We now get $u_1 = 7, u_2 = 5, u_3 = 0, v_1 = 5, v_2 = 16, v_3 = 13, v_4 = 16, v_5 = 0$. This is not optimal since $v_4 - u_1 = 9 > 8$. Thus we add to this square (40 is the maximum!), and adjust the three included squares that form a square with it, to give the following allocation

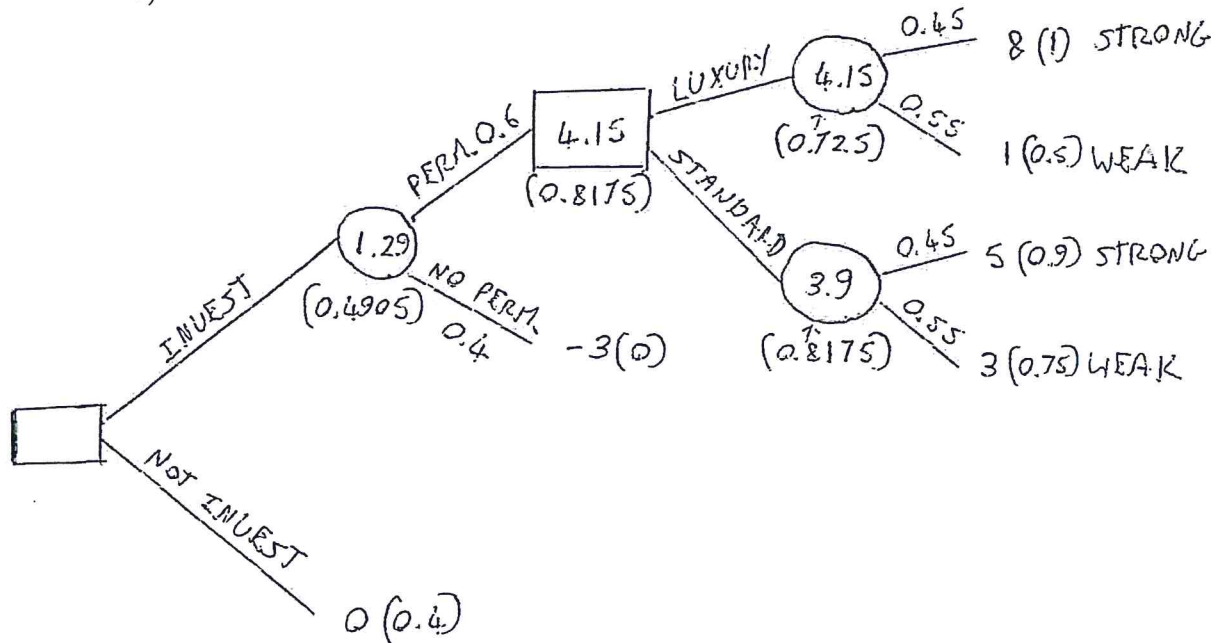
	1	2	3	4	5	
A	15	9	6	8 40	0	40
B	9	11 50	8 10	13	0	100
C	5 30	17	13 40	16 20	0 10	60
	30	50	50	60	10	

$u_1 = 7, u_2 = 5, u_3 = 0, v_1 = 5, v_2 = 16, v_3 = 13, v_4 = 16, v_5 = 0$, which comparing to the vacant squares gives an optimal allocation, with cost 1940.

4

3

a)



b) If planning permission is obtained and they choose a Luxury development, they gain $8 \times 0.45 + 1 \times 0.55 = 4.15$ million pounds. Similarly with a Standard development they gain $5 \times 0.45 + 3 \times 0.55 = 3.9$. Thus Luxury is best, assuming planning permission is obtained. If the investment is made the expected profit is thus $0.6 \times 4.15 + 0.4 \times -3 = 1.29 > 0$. Thus the company should invest, and go for Luxury development if planning permission is obtained.

c) Denote the probability of strong demand by p . Then if planning permission is obtained and they choose a Luxury development, they gain $8 \times p + 1 \times (1-p) = 1 + 7p$ million pounds. Similarly with a Standard development they gain $5 \times p + 3 \times (1-p) = 3 + 2p$. Thus Luxury is best if $1 + 7p > 3 + 2p \Rightarrow p > 0.4$, assuming planning permission is obtained. If the investment is made the expected profit is thus $0.6 \times \max(1 + 7p, 3 + 2p) + 0.4 \times -3 = \max(0.42p - 0.6, 1.2p + 0.6)$. This maximum is clearly greater than 0, since one of its two terms is greater than 0. Thus the company should invest, and go for Luxury (Standard) development if $p > 0.4$ ($p < 0.4$) and planning permission is obtained.

d) Now the probability of planning permission being granted and strong demand occurring is $0.45 \times 0.7 = 0.315$, and the probability of permission being granted and weak demand occurring is $0.55 \times 0.4 = 0.22$. The probability of permission not being granted is thus $1 - 0.315 - 0.22 = 0.465$. Thus if investment and Luxury development is chosen the expected profit is $8 \times 0.315 + 1 \times 0.22 - 3 \times 0.465 =$

1.345 and if investment and Standard development is chosen the expected profit is $5 \times 0.315 + 3 \times 0.22 - 3 \times 0.465 = 0.84$. 1.345 is larger than 0.84 and 0, thus investment and Luxury development should be chosen.

e) We simply replace the monetary returns by their utilities in the above calculations. Thus if planning permission is obtained the expected utility for Luxury is $1 \times 0.45 + 0.5 \times 0.55 = 0.725$ and for Standard is $0.9 \times 0.45 + 0.75 \times 0.55 = 0.8175$. Thus Standard development is now the best policy. If investment is made the expected utility is $0.6 \times 0.8175 + 0.4 \times 0 = 0.4905 > 0.4$ which is the utility of 0. Thus the company should invest, and go for Standard development if planning permission is obtained.

4

a) Maximise $100x_1 + 240x_2 + 300x_3 + 400x_4$
 subject to $24x_1 + 40x_2 + 30x_3 + 60x_4 \leq 2400$,
 $2x_1 + 5x_2 + 10x_3 + 10x_4 \leq 500$,
 $x_1, x_2, x_3, x_4 \geq 0$.

b)

	x_1	x_2	x_3	x_4	
y_1	24	40	30	60	2400
y_2	2	5	10	10	500
	-100	-240	-300	-400	0

→

	x_1	y_1	x_3	x_4	
x_2	0.6	0.025	0.75	1.5	60
y_2	-1	-0.125	6.25	2.5	200
	44	6	-120	-40	14400

→

	x_1	y_1	y_2	x_4	
x_2			-0.12		36
x_3	-0.16	-0.02	0.16	0.4	32
	24.8	3.6	19.2	8	18240

Thus the optimal solution is $x_2 = 36$ Necklaces and $x_3 = 32$ Bracelets with profit £18240.

c) We now require in addition the condition $x_1 + x_2 + x_3 + x_4 \leq 65$, which leads to

	x_1	x_2	x_3	x_4	
y_1	24	40	30	60	2400
y_2	2	5	10	10	500
y_3	1	1	1	1	65
	-100	-240	-300	-400	0

It is now logical to choose the final column, as this extra severe restriction is likely to favour the increase of the most profitasble items. This gives

	x_1	x_2	x_3	y_1	
x_4	0.4	$2/3$	0.5	$1/60$	40
y_2	-2	$-5/3$	5	$-1/6$	100 →
y_3	0.6	$1/3$	0.5	$-1/60$	25
	60	$80/3$	-100	$20/3$	16000

	x_1	x_2	y_2	y_1	
x_4	0.6	$5/6$	-0.1	$1/30$	30
x_3	-0.4	$-1/3$	$1/5$	$-1/30$	20 →
y_3	0.8	0.5	-0.1	0	15
	20	$-20/3$	20	$10/3$	18000

	x_1	y_3	y_2	y_1	
x_4		$-5/3$			5
x_3		$2/3$			30
x_2	1.6	2	-0.2	0	30
	$92/3$	$40/3$	$56/3$	$10/3$	18200

Thus the new optimal solution is $x_2 = 30$ Necklaces, $x_3 = 30$ Bracelets and $x_4 = 5$ Figurines with profit £18200.